1. General

The Allocation module is part of the aNySim DP-system:

A generic thruster allocation has been developed that allocates main thrusters, azimuthing thrusters, bow thrusters, tunnel thrusters and rudders. The allocation is based on a minimisation of the total squared thrust such that the total thruster and rudder forces equal the forces which are required by the controller.

Use is made of the Lagrange multiplier method to find an optimal solution. Applying the Lagrange multiplier method to this particular optimisation problem leads to a non-linear system. This system is solved in an iterative way with Newton’s method.
The allocation considers an unlimited number of azimuth thrusters, bow tunnel thrusters, stern tunnel thrusters and main propellers. Rudders may be placed behind a main propeller. A maximum of two rudders is possible. It is assumed that a rudder is placed directly behind a main thruster (i.e. its longitudinal position is equal to that of the main propeller).

The positions of thrusters and rudders are given in a 2D co-ordinate system (the vertical position of the thrusters and rudders is not considered; only the horizontal forces and moment are considered). The co-ordinate system has its origin at the vessel's CoG. The required and allocated moments are considered w.r.t. this origin.

The following abbreviations are used:

- number of azimuthing thrusters: $N_{azim}$
- number of bow tunnel thrusters: $N_{bow}$
- number of stern tunnel thrusters: $N_{stern}$
- number of main propellers: $N_{main}$
- total number of thrusters: $N_{total}$
- number of rudders: $N_{rud}$
- number of fixed azimuthing thrusters: $N_{fixed}$
- number of unknowns: $N_{unknown}$
- fixed azimuth angle $i$: $\text{fixedangle}_i$
- x-co-ordinate of thruster $i$: $x_i$
- y-co-ordinate of thruster $i$: $y_i$
- force of thruster $i$: $T_i$
- x-force of thruster $i$: $T_{x_i}$
- y-force of thruster $i$: $T_{y_i}$
- x-force of rudder $i$: $F_{xrud_i}$
- y-force of rudder $i$: $F_{y rud_i}$
- maximum force of thruster $i$: $T_{max_i}$
- maximum of all maximum thruster forces: $T_{max}$
- angle of rudder $i$: $\text{rudang}_i$
- maximum rudder angle of rudder $i$: $\text{maxangle}_i$
- minimum rudder angle of rudder $i$: $\text{minangle}_i$
- height of rudder $i$: $\text{rudheight}_i$
- width of rudder $i$: $\text{rudwidth}_i$
- fixed rudder angle $i$: $\text{fixedrudangle}_i$
2. Objective function

The unknowns that have to be solved are the thrust of each thruster, the angle of the azimuthing thruster and the angle of the rudders. Instead of solving the azimuthing thruster force and the azimuth angle, the forces in the x-direction and y-direction are solved. The total vector with unknowns looks as follows:

\[ \bar{x} = \begin{pmatrix} T_{x_1} \\ T_{y_1} \\ \vdots \\ T_{y_{\text{Nazim}}} \\ T_{\text{Nazim}+1} \\ \vdots \\ \vdots \\ T_{\text{Nazim}+\text{Nbow}+\text{Nstern}+\text{Nmain}} \end{pmatrix} \]

The number of unknowns is thus \( N_{\text{unknown}} = 2N_{\text{azim}} + N_{\text{main}} + N_{\text{bow}} + N_{\text{stern}} + N_{\text{rud}} \)

The allocation is based on a minimisation of the sum of the total squared thrust. The following function is minimised (referred to as objective function):

\[
F(\bar{x}) = \sum_{i=1}^{N_{\text{max}}} \frac{T_i}{T_{\text{max}}}^{\text{power}} + \sum_{i=1}^{N_{\text{max}}} w_i \left( \frac{T_i}{T_{\text{max}}_i} - 1 \right)^2
\]

The last summation in the objective function is a penalty function that adds to the objective function when the maximum thrust of a thruster is exceeded (saturation). \( w_i \) is a weight factor which is defined as follows:

\[
w_i = \begin{cases} c_i & \text{if } T_i > T_{\text{max}}_i \\ 0 & \text{else} \end{cases}
\]

This makes sure the penalty function only adds up to the objective function when the maximum thruster force is exceeded. Instead of further increasing the thruster force of a
saturated thruster, the force of other available thrusters is increased first. Tests with the algorithm showed that the following factor is appropriate to use:

\[ c_i = 10000 \]

The value of the constant 'power' should be larger than one. Often used values are power=1.5 and power=2. In aNySim it is set at 2.

3. Constraints

The thrusters have to be allocated such that several constraints are satisfied. At first, the total thruster and rudder force should equal the required forces. This leads to three constraints for the force in x-direction, the force in y-direction and the yaw moment.

\[
g_1 = F_x - \sum_{i=1}^{\text{Nazim}} T_{x_i} - \sum_{i=\text{Nazim}+1}^{\text{Nazim+nmain}} T_{i} - \sum_{i=1}^{\text{Nrud}} F_{xrud_i} = 0
\]

\[
g_2 = F_y - \sum_{i=1}^{\text{Nazim}} T_{y_i} - \sum_{i=\text{Nazim}+1}^{\text{Nazim+nmain}} T_{i} - \sum_{i=1}^{\text{Nrud}} F_{yrud_i} = 0
\]

\[
g_3 = M_z - \sum_{i=1}^{\text{Nazim}} (x_i T_{y_i} - y_i T_{x_i}) - \sum_{i=\text{Nazim}+1}^{\text{Nazim+nmain}} (-y_i T_{i}) - \sum_{i=1}^{\text{Nrud}} \sum_{i=\text{Nmain+nstern}}^{\text{Nmain+nmain+1}} F_{yrud_i x_i} - \sum_{i=1}^{\text{Nrud}} \sum_{i=1}^{\text{Nmain+nstern}} F_{xrud_i y_i} = 0
\]

The rudder forces depend on the amount of thrust generated by the main propeller behind which it is located. The following simplified rudder model is used:

\[
F_{x_{\text{rud}}} = -\frac{8}{2\pi} \text{SR} (1.69 - 1.51C_B)^2 T_{\text{main}} \frac{H_{\text{rud}}^2}{\text{SR}^2} \sin^2(\delta) (C_{di} \cos^4(\delta) + C_{dy} \sin(\delta))
\]

\[
F_{y_{\text{rud}}} = \frac{8}{2\pi} \text{SR} (1.69 - 1.51C_B)^2 T_{\text{main}} \frac{H_{\text{rud}}^2}{\text{SR}^2} \sin(\delta) \cos(\delta) (C_{ld} + C_{dy} \sin(\delta) - C_{di} \sin^2(\delta) \cos^2(\delta))
\]

where:

\[ H_{\text{rud}} = \text{rudder height} \]

\[ W_{\text{rud}} = \text{rudder width} \]

\[ T_{\text{main}} = \text{thrust main propeller} \]

\[ \delta = \text{rudder angle} \]

\[ \text{SR} = H_{\text{rud}} \times W_{\text{rud}} \]

\[ C_{dy} = 2.3 \]

\[ C_{di} = \frac{C_{ld}^2}{0.9\pi A_{eir}} \]
CLd = \frac{6.13 \cdot \text{Aer}}{\text{Aer} + 2.25}

\text{Aer} = \frac{2 \cdot H_{\text{nd}}}{W_{\text{nd}}}

C_B = \text{block coefficient}

In this formulation it is assumed that the rudder angle is fixed. The actual optimisation with respect to rudder angles takes place in a loop around the thrust allocation (for a set of different rudder angles the optimum thruster allocation is computed. The rudder angle is chosen that leads to the lowest objective function)

Additional constraints may be imposed by fixing the angle of an azimuthing thruster. For each of the azimuthing thrusters which is fixed (total of N_{\text{fixed}}), a constraint holds:

\begin{align*}
g_{3-j} = -T_x j_i \sin(\text{fixedangle}_j) + T_y j_i \cos(\text{fixedangle}_j) \\
j = 1 \ldots N_{\text{fixed}}
\end{align*}

When the vessel is equipped with two rudders, the method has the possibility for various types of rudder allocations, namely:

1) Fixed rudders. The rudder angles are fixed.
2) Tandem rudder allocation. The rudder angles of the two rudders are the same.
3) Independent rudder allocation. The rudder angles are allocated independently.

This does not lead to extra constraints in the thruster allocation since the optimisation with respect to the rudder angle takes place in a loop around the thrust allocation algorithm.

4. Lagrange multiplier method

The allocation of the thrusters has been reduced to minimising an objective function with several constraints. As shown above, the number of constraints depends amongst others on the number of azimuthing thrusters with fixed angles. In general form, this can be written as:

\text{minimise } F(\bar{x})

with the constraints

\begin{align*}
g_1(\bar{x}) &= 0 \\
\vdots \\
g_N(\bar{x}) &= 0
\end{align*}
where N is the number of constraints and \( \hat{x} \) the vector with variables to be allocated (see section objective function, there are \( N_{\text{unknown}} \) variables).

According to the Lagrange multiplier theorem, this is equivalent with solving the system of equations:

\[
\begin{align*}
g_1(\hat{x}) &= 0 \\
&\vdots \\
g_N(\hat{x}) &= 0
\end{align*}
\]

\[
\begin{align*}
\frac{\partial F}{\partial x_1} + \lambda_1 \frac{\partial g_1}{\partial x_1} + \lambda_2 \frac{\partial g_2}{\partial x_2} + \ldots + \lambda_N \frac{\partial g_N}{\partial x_N} &= 0 \\
\frac{\partial F}{\partial x_2} + \lambda_1 \frac{\partial g_1}{\partial x_1} + \lambda_2 \frac{\partial g_2}{\partial x_2} + \ldots + \lambda_N \frac{\partial g_N}{\partial x_N} &= 0 \\
&\vdots \\
\frac{\partial F}{\partial x_{\text{unknown}}} + \lambda_1 \frac{\partial g_1}{\partial x_{\text{unknown}}} + \lambda_2 \frac{\partial g_2}{\partial x_{\text{unknown}}} + \ldots + \lambda_N \frac{\partial g_N}{\partial x_{\text{unknown}}} &= 0
\end{align*}
\]

where \( \lambda_1\ldots\lambda_N \) are the Lagrange multipliers. This is a system with \( N+N_{\text{unknown}} \) equations for the \( N+N_{\text{unknown}} \) variables. However, the system is non-linear when penalty functions are added to the objective function (saturated thrusters).

Therefore, the system of equations has to be solved in an iterative way.

5. Newton’s method

The system of equations to be solved can be written as

\[
\tilde{H}(\tilde{x}) = 0
\]

where \( \tilde{H}(\tilde{x}) \) is a vector function. \( \tilde{x} \) is a vector consisting of the vector with unknowns, \( \hat{x} \), and a vector with the Lagrange multipliers, \( \lambda \):

\[
\tilde{x} = \begin{pmatrix} \hat{x} \\ \lambda \end{pmatrix}
\]
According to the method of Newton, for a suitable first iteration (guess of solution) a solution can be found with the following iterative scheme:

$$\tilde{x}_{k+1} = \tilde{x}_k - \tilde{H} \frac{\tilde{x}_k - \tilde{H} \tilde{x}_k}{\nabla H \tilde{x}_k}$$

This scheme is quadratically convergent. $\nabla \tilde{H}$ is the Jacobian of the vector function:

$$\nabla \tilde{H} = \begin{pmatrix} \frac{\partial H_1}{\partial x_1} & \ldots & \frac{\partial H_1}{\partial x_{N+unknown}} \\ \vdots & \ddots & \vdots \\ \frac{\partial H_{N+unknown}}{\partial x_1} & \ldots & \frac{\partial H_{N+unknown}}{\partial x_{N+unknown}} \end{pmatrix}$$

The derivatives in the Jacobian are computed with finite differences (central difference scheme).

The method requires a first estimate of the solution. The following first estimate is made:

1) All thrusters are set to 90 percent of their maximum thrust
2) Main propellers, bow tunnel thrusters and stern tunnel thrusters generate thrust in the same X- and Y-direction as the required force
3) Azimuthing thrusters are either set to their fixed angle or to the angle of the required force vector
4) Lagrange multipliers are set to zero

### 6. Forbidden zones

The algorithm allows for forbidden zones of azimuthing thrusters. The procedure is as follows:

1) The allocation is carried out without forbidden zone restrictions
2) It is checked whether any of the azimuth angles are located inside a forbidden zone. If not, the algorithm ends
3) If an azimuth angle is located inside a forbidden zone, the azimuth angle is fixed to the nearest boundary (upper or lower) of the forbidden zone. In case it was already at a boundary the azimuth angle is only set to the other boundary when the distance between the required azimuth angle and the other boundary is 3 times smaller than the distance to the boundary where the azimuthing thruster already is. A re-allocation is carried out with the azimuth angles fixed. However, this may lead to other azimuthing thrusters being located in their forbidden zone. Therefore, this
procedure is repeated until all azimuth angles are outside or on the boundary of the forbidden zones.

7. Rudder allocation

Rudder allocation takes place in a loop around the thrust allocation. There are three types of rudder allocation:

1) Fixed rudders. The rudder angles are fixed.
2) Tandem rudder allocation. The rudder angles of the two rudders are the same.
3) Independent rudder allocation. The rudder angles are allocated independently.

The first type of allocation is the easiest one. The rudder(s) is (are) fixed to their angle and a single thrust allocation takes place.

The second and third type of allocation require more CPU-time. The thrust allocation is repeated for a set of rudder angles. With the tandem option this is a single set of $N_{\text{rudangle}}$, with the independent option all combinations of $N_{\text{rudangle}} \times N_{\text{rudangle}}$ are evaluated. Finally, the rudder angles which give the lowest objective function value are selected.